

An Adaptive Entanglement Distillation Scheme Using Quantum Low Density Parity Check Codes

K. H. Ho and H. F. Chau

Department of Physics and Center of Theoretical and Computational Physics,
The University of Hong Kong, Pokfulam Road, Hong Kong

Abstract—Quantum low density parity check (QLDPC) codes are useful primitives for quantum information processing because they can be encoded and decoded efficiently. Besides, the error correcting capability of a few QLDPC codes exceeds the quantum Gilbert-Varshamov bound. [1] Here, we report a numerical performance analysis of an adaptive entanglement distillation scheme using QLDPC codes. In particular, we find that the expected yield of our adaptive distillation scheme to combat depolarization errors exceed that of Leung and Shor [2], [3] whenever the error probability is less than about 0.07 or greater than about 0.28. This finding illustrates the effectiveness of using QLDPC codes in entanglement distillation.

Index Terms—Adaptive Algorithm, Depolarization Error, Entanglement Distillation, Quantum Low Density Parity Check Code

I. INTRODUCTION

Armed with quantum computers, Alice and Bob want to share copies of high fidelity Bell state through an unknown noisy quantum channel. One way to do so is to compare their measured error syndromes of their shares of the quantum particles using a pre-determined quantum error-correcting code (QECC) and then to perform the necessary error recoveries. Thanks to the quantum Gilbert-Varshamov bound, there exists a QECC that produces perfect copies of Bell state provided that the quantum error rate of the noisy channel is less than about 19%. However, finding such a QECC as well as executing the corresponding decoder can both be computationally intractable. Another way to share copies of high fidelity Bell state is to apply entanglement distillation purification (EDP) such as the recurrence method [4], [5]. More precisely, by two-way local operations and classical communications (LOCC2), Alice and Bob discard those particles whose measurement results are not consistent with that of the corresponding Bell states. Thus, two-way EDP can be regarded as a carefully designed quantum-error-detection-code-based error rejection method. It can tolerate a higher error level than any one-way QECC-based method at the expense of having a much lower yield. [4], [5], [6]

QECC- and EDP-based entanglement distillation methods can be extended in many ways. For instance, Gottesman and Lo [7] as well as Chau [8] introduced adaptive schemes using both QECCs and EDPs to distill copies of almost perfect EPR pairs. Their schemes increase the error tolerance level at the expense of the yield. Along a different line, Vollbrecht and Verstraete [9] as well as Hostens *et al.* [10] generalized the recurrence method to raise the yield of entanglement distilla-

tion. Recently, Leung and Shor [3] introduced an entanglement distillation protocol extending the earlier works of Maneva and Smolin [11] as well as Leung and Shor [2]. Specifically, Leung and Shor [3] used a carefully constructed adaptive EDP-based protocol with universal hashing to increase the yield over a certain range of channel error rates.

It is instructive to find a way for Alice and Bob to share Bell states with an even higher yield without sacrificing their fidelity too much. Naively, Alice and Bob may optimize the yield by estimating the noise level of the quantum channel before choosing the appropriate method. However, this method is not ideal as the noise level of the quantum channel may change, say, in the presence of an adversary.

In view of the similarity between QECC-based and EDP-based schemes, it is instructive to study methods that estimate the error rate and perform the necessary error recovery or error rejection simultaneously. Let us use the following setting as the basis of our investigation. Alice and Bob pick a QECC. They compare their error syndrome measurements and use them to decide which qubits have to be discarded and which have to be error-corrected. However, not every QECC \mathcal{C} is suitable for this purpose because the error-correcting capability of a typical subcode of \mathcal{C} formed by puncturing a few discarded qubits may be drastically reduced. Fortunately, quantum low density parity check (QLDPC) code is ideal for this job. First, owing to the fact that QLDPC codes have sparse parity check matrices, their average error-correcting capabilities do not in general change greatly with the deletion of a few qubits. Second, QLDPC codes can be efficiently constructed [1], [12]. Third, efficient approximate decoding methods such as belief propagation for classical low density parity check codes (or LDPC codes for short) [13], [14], [15] can be readily extended to QLDPC codes. Therefore, it makes sense to investigate the performance of a QLDPC-code-based entanglement distillation scheme. In fact, a preliminary study along this line by one group has been reported in [16].

In Sec. II, we briefly review the existing literature of LDPC, QLDPC as well as the construction of the EDP from QECC. In Sec. III, we introduce a QLDPC-code-based entanglement distillation scheme with LOCC2 as well as the performance indicators in our analysis. Then we study the performance of our scheme to combat depolarization errors numerically in Sec. IV. In particular, we find that our scheme has a better yield than the recent method by Leung and Shor [2], [3] to combat depolarization errors whenever the error probability is less than about 0.07 or greater than about 0.28. Finally, we

conclude by giving the reasons why our scheme has a high yield in Sec. V. We also suggest some possible future works on QLDPC-code-based adaptive EDP there.

II. PRELIMINARIES AND PRIOR ARTS

A. Classical low density parity codes

Definition 1: A classical low density parity check (LDPC) code is a linear block code over a finite field $GF(q)$ that has a sparse parity check matrix. In particular, a (d_v, d_c) -regular LDPC code has a sparse parity check matrix H with d_v non-zero entries in each column and d_c non-zero entries in each row. [15], [17], [18], [19]

LDPC code can be represented by the so-called Tanner graph. Recall that a Tanner graph of a linear code \mathcal{C} with a parity check matrix H is a bipartite graph with vertex set $V = V_1 \cup V_2$. Each variable node in V_1 is associated with a bit of the code represented by a column of H ; and each check node in V_2 is associated with a generator of the code represented by a row of H . There is an edge linking $i \in V_2$ and $j \in V_1$ if and only if $H_{ij} \neq 0$.

Let \mathcal{C} be a LDPC code encoding k bits of information as an n -bit string by a sparse parity check matrix H . We denote the encoded message by the column vector \mathbf{t} . After passing this encoded message through a noisy channel, the receiver gets the column vector $\mathbf{r} = \mathbf{t} + \mathbf{e}$ where \mathbf{e} is what we called the noise vector. The task of a decoder, therefore, is to infer \mathbf{x} given the error syndrome $\mathbf{s} = H\mathbf{r}$ and the assumed properties of the channel. More precisely, the decoder returns a column vector \mathbf{x} that maximizes the posterior conditional probability $\Pr(\mathbf{x}|\mathbf{s}, H)$ of finding \mathbf{x} given the syndrome \mathbf{s} and the parity check matrix H .

Many efficient approximate decoding strategies for LDPC codes can be regarded as message passing algorithms executed on the corresponding Tanner graph. Message-passing decoding generally begins with variable nodes sending messages to their neighboring check nodes. Decoding continues with rounds of messages sending back and forth between the check nodes and the variable nodes; and new messages are computed by each node as functions of messages previously sent to them. The decoding algorithm terminates if a tentative decoding is found.

Famous for its linear runtime in the codeword size n , belief propagation is one of the most commonly used message passing algorithm in which the messages passed between nodes in a Tanner graph are conditional probabilities [13], [14], [15]. More importantly, we shall show in Sec. III that belief propagation algorithm is applicable to quantum stabilizer codes whose generators of the stabilizer is sparse.

B. Quantum low density parity check codes

Definition 2: A quantum low density parity check (QLDPC) code is a quantum stabilizer block error-correcting code over a finite field $GF(q)$ that has a sparse parity check matrix. In particular, a (d_v, d_c) -regular QLDPC code has a sparse parity check matrix H with a constant column weight d_v and a constant row weight d_c . [1], [12]

For example, the quantum error-detection code associated with each round of entanglement distillation by the recurrence method and the Leung and Shor method are (1, 2)- and (2, 4)-regular QLDPC codes, respectively. (In some sense, these two codes are atypical QLDPC codes as they compose of tensor products of block codes of sizes 2 and 4, respectively.) Actually, a large number of QLDPC codes exist for a sufficiently large block code size n . Existing ways to construct them include:

1) *Dual-containing CSS codes:* MacKay *et al.* [1] constructed a few $GF(4)$ QLDPC codes from Calderbank-Shor-Steane (CSS) codes [20], [21] with certain constraints on the global structure of their parity check matrices. They include:

- (i) *Bicycle codes:* To construct a $[n, n - k]$ bicycle $GF(4)$ QLDPC CSS code with row weight d_c , MacKay *et al.* first selected a random $(n/2) \times (n/2)$ cyclic binary sparse-matrix C_B with row weight $d_c/2$ and defined a $(n/2) \times n$ matrix H by

$$H = [C_B, C_B^T], \quad (1)$$

where C_B^T is the transpose of C_B . Then, they deleted some rows from H to obtain a new matrix H_B with k rows. One can easily check that H_B is self-dual in the sense that $H_B H_B^T = 0$. As a result, H_B can be used to construct a $[n, n - 2k]$ binary CSS code. MacKay *et al.* further showed that the performance of some bicycle codes is better than the Gilbert-Varshamov rate for binary CSS codes [1].

- (ii) *Uicycle codes:* A unicycle $GF(4)$ code is constructed by making use of a perfect difference set over an additive group. All pairs of rows of the cyclic binary matrix C_U that make from the perfect difference set have an overlap of one. To make the matrix self-dual, a new column of all ones is appended to the matrix C_U . Thus, every pair of distinct rows of the resultant matrix H_U have even overlapping. The self-dual matrix H_U can then be used to construct a CSS-type QLDPC code [1].

2) *Group theoretical construction:* Instead of using CSS codes as the starting point, Camara *et al.* constructed QLDPC codes by selecting low weight generators of the stabilizer using certain group theoretical method [12]. Numerical simulations of the performance of a (4, 8)- and a (6, 12)-regular QLDPC codes using their group theoretical method on the depolarizing channel can be found in Ref. [12].

3) *Quantum quasi-cyclic LDPC codes:* Hagiwara and Imai invented a CSS-type construction of quantum quasi-cyclic LDPC code. Their construction is based on algebraic combinatorics, and the performance of their codes was analyzed in Ref. [22]. Recently, Hsieh *et al.* proposed and investigated a new type of QLDPC codes from classical quasi-cyclic LDPC codes [23].

4) *QLDPC codes constructed from finite geometries:* Aly proposed and analyzed the performance of a class of QLDPC whose parity check matrix are adapted to be self-orthogonal with containing only one cycle of length four. [24]

5) *Asymmetric QLDPC codes*: Sarvepalli *et al.* constructed a CSS-type of asymmetric QLDPC based on BCH and finite geometry LDPC codes to take account the asymmetry for the occurrence of bit flip and phase flip errors. [25]

C. Entanglement distillation with two-way classical communications by quantum error-correcting codes

The general procedure of an adaptive LOCC2 stabilizer-code-based entanglement distillation purification (EDP2) protocol can be described below [7], [26]. Alice prepares n Bell states and sends the second halves to Bob. Alice and Bob measure up to $(n - k)$ commuting generators of the stabilizer code one by one. After measuring each generator, they may throw away some of their shared quantum particles upon comparing their measurement results. Then they compute the error syndrome. The $(i + 1)$ th generator used may depend on the results obtained in the first i measurements as long as this generator commutes with all the previously measured operators. In the last step of the protocol, Alice and Bob perform local unitary transformation based on their earlier measurement results to distill the k almost perfect Bell states. Note that all one-way QECC-based entanglement purification schemes together with the two-way recurrence method introduced by Bennett *et al.* [4], [5] and its various extensions [2], [9], [10], [11], [27] can all be regarded as special cases of this EDP2 protocol.

III. AN ADAPTIVE ENTANGLEMENT DISTILLATION PROTOCOL USING QUANTUM LOW DENSITY PARITY CHECK CODES

A. Dual-containing quantum low density parity check stabilizer codes

Among the existing QLDPC code construction schemes in the literature, some can only build CSS codes and some may not be efficient. So, in order to increase the error-tolerant capacity of our practical adaptive entanglement distillation protocol, we have to find a simple and efficient way to construct a large number of QLDPC stabilizer codes. Actually, our QLDPC code construction works for any q -ary code where $q = p^n$ is a prime power. We follow the notation of Ashikhmin and Knill [28] by defining the unitary operators X_a and Z_b acting on a q -dimensional Hilbert space by

$$X_a : |i\rangle \mapsto |i + a\rangle \quad (2)$$

and

$$Z_b : |i\rangle \mapsto \varpi_p^{\text{Tr}(ib)} |i\rangle \quad (3)$$

for all $a, b, i \in GF(q)$, where ϖ_p is the p th root of unity and

$$\text{Tr}(i) = i + i^p + \dots + i^{p^{n-1}} \in GF(p) \quad (4)$$

is the absolute trace of $i \in GF(q)$. Note that all arithmetic inside the state ket and in the exponent of ϖ_p is performed in the finite field $GF(q)$. We also identify the unitary operator $X_a Z_b$ with $a + b\omega_{q^2} \in GF(q^2)$ where ω_{q^2} is a fixed primitive element in $GF(q^2)$. Using this identification, we may abuse our language by saying, for example, that a qubit has experienced an error $a + b\omega_{q^2}$.

Our QLDPC stabilizer code construction is an extension of the bicycle QLDPC CSS code construction by MacKay *et al.* [1] reviewed in Sec. II. And it comes from a simple but important observation concerning the matrix H in Eq. (1). Suppose the elements of the matrix C_B satisfy $(C_B)_{i,j} \equiv (C_B)_{ij} = \alpha_{i-j}$ for some $\alpha_{i-j} \in GF(q^2)$. So, for $1 \leq i, i', j \leq n/2$,

$$(C_B)_{i,i'+j}^T = (C_B)_{i',j} \quad (5)$$

and

$$(C_B)_{i',i'+j}^T = (C_B)_{i,j}. \quad (6)$$

Then, rows of the bicyclic matrix $H = [C_B, C_B^T]$ are mutually orthogonal to each other with respect to the skew-symmetric inner product

$$(a + b\omega_{q^2}|c + d\omega_{q^2}) \equiv \text{Tr}(ad - bc) \in GF(p) \quad (7)$$

for all $a, b, c, d \in GF(q)$, irrespective of whether C_B is sparse or not. Since

$$X_c Z_d X_a Z_b = \varpi_p^{(a+b\omega_{q^2}|c+d\omega_{q^2})} X_a Z_b X_c Z_d, \quad (8)$$

the rows of H can be identified as the generators of the stabilizer of a q -ary QECC [28], [29]; and so is H_B , the matrix obtained by deleting a few rows of H .

In particular, by choosing $(\alpha_i)_{i=1}^{n/2}$ to be a sparse vector whose elements are in $GF(q^2)$ so that C_B is a sparse $(n/2) \times (n/2)$ matrix, H_B becomes the parity check matrix of a q -ary QLDPC code. More importantly, the $GF(q^2)$ QLDPC code constructed in this way is not necessarily a CSS code.

Interestingly, we may build a large number of regular QLDPC codes using this modified bicycle construction. The trick is to pick the sparse vector $(\alpha_i)_{i=1}^{n/2}$ in such a way that

$$|\{i : \alpha_{n'+i+j}\}| = u \quad (9)$$

for all j with the constraint $(n/2)$ is divisible by n' . Then it is easy to check that the parity check matrix H constructed is $(n'u, 2n'u)$ -regular. And, by deleting the $(in' + j)$ th row of H for $i \in \mathbb{N}$, $j \in J$ where J is a proper subset of $\{1, 2, \dots, n'\}$, the resultant parity check matrix H_B corresponds to a $([n' - |J|]u, 2n'u)$ -regular q -ray QLDPC code. For instance, let $q = 2, n = 12, n' = 3, (\alpha_i) = (1, \omega_4, \omega_4^2, 0, 0, 0)$ where ω_4 is a primitive element in $GF(4)$, and $J = \{3\}$. Then our construction gives the $(2, 6)$ -regular binary QLDPC stabilizer (but non-CSS) code

$$\left[\begin{array}{cccccc|cccccc} 1 & \omega_4 & \omega_4^2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \omega_4^2 & \omega_4 \\ 0 & 1 & \omega_4 & \omega_4^2 & 0 & 0 & \omega_4 & 1 & 0 & 0 & 0 & \omega_4^2 \\ 0 & 0 & 0 & 1 & \omega_4 & \omega_4^2 & 0 & \omega_4^2 & \omega_4 & 1 & 0 & 0 \\ \omega_4^2 & 0 & 0 & 0 & 1 & \omega_4 & 0 & 0 & \omega_4^2 & \omega_4 & 1 & 0 \end{array} \right]. \quad (10)$$

It is easy to check that the (quantum) rate of the (d_v, d_c) -regular QLDPC code constructed in this way is greater than or equal to $1 - d_v/d_c$, where the equality holds if and only if the rows of H are linearly independent over $GF(q)$. In our subsequent study, we only consider those H 's with full rank so that their rate is equal to $1 - d_v/d_c$. Surely, this extra constraint on the choice of H is not very restrictive as our construction is likely to give H with full rank anyway.

Note that for a typical sparse vector $(\alpha_i)_{i=1}^{n/2}$ satisfying Eq. (9), the number $|\{i : \alpha_i = \beta\}|/(n/2)$ is about the same for all $\beta \in GF(q^2)^*$. To summarize, we have succeeded in constructing a large number of regular q -ary QLDPC codes. The construction is very efficient. Besides, their regularity and almost equal probability of occurrence of non-zero elements in $(\alpha_i)_{i=1}^{n/2}$ make them reasonably effective to combat quantum errors.

B. Belief propagation algorithm for quantum stabilizer codes

Belief propagation algorithm for classical LDPC codes can be extended to the case of stabilizer code as follows. (See also Ref. [30] for a description of a similar belief algorithm applied to graph states.) A stabilizer code \mathcal{C} associated with a parity check matrix H can be represented by a Tanner graph with vertex set $V = V_1 \cup V_2$. Each variable node in V_1 is associated with a qubit of the code represented by a column of H ; and each check node in V_2 is associated with a generator of the code represented by a row of H . There is an edge linking $i \in V_2$ and $j \in V_1$ if only if $H_{ij} \neq 0$.

By passing the messages between the nodes, the task of the belief propagation decoding algorithm is to infer a tentative decoding \tilde{x} . That is to say, \tilde{x} is the most likely value of error experienced by the shared EPR pairs based on the measured error syndrome vector

$$\mathbf{s} \equiv (s_i)_{i \in V_2} = \left(\sum_{j \in V_1} (H_{ij} | e_j) \right)_{i \in V_2}, \quad (11)$$

where the check node $s_i \in GF(p)$ is the i th component of the syndrome \mathbf{s} and e_j is the error experienced by the variable node x_j . We call $\mathbf{e} \equiv (e_j)_{j \in V_1}$ the noise vector of the state shared by the sender and the receiver.

The messages consist of two types of conditional probabilities Q_{ij}^α and R_{ij}^α associated with each non-zero entry in the parity check matrix H for all $\alpha \in GF(q^2)$. To aid discussions, we call the j th component of the tentative decoding vector \tilde{x} the variable node $\tilde{x}_j \in GF(q^2)$. The quality Q_{ij}^α approximates the belief that the qubit \tilde{x}_j has experienced the error $\alpha \in GF(q^2)$ given the messages received from all its checks other than i . And the quality R_{ij}^α is the probability of check i being satisfied given that the variable node \tilde{x}_j has experienced an error in the state $\alpha \in GF(q^2)$ and the components of \tilde{x} other than \tilde{x}_j have a separable distribution given by the probabilities Q_{ij}^α 's.

Initially, each message Q_{ij}^α is set to the prior probability f_j^α that x_j has experienced an error α . In situation of our interest, f_j^α is a quality of the quantum channel linking the two parties who would like to perform entanglement distillation. In each step, the quantities R_{ij}^α are updated according to the equation

$$R_{ij}^\alpha = \sum_{\mathbf{x}' : x'_j = \alpha} \left[\Pr(s_i | \mathbf{x}') \prod_{j' \in \mathcal{N}(i) \setminus \{j\}} Q_{ij'}^{x'_{j'}} \right], \quad (12)$$

where $\mathcal{N}(i) \equiv \{j : H_{ij} \neq 0\}$ denotes the set of variable nodes participated in the check i and

$$\Pr(s_i | \mathbf{x}') = \begin{cases} 1 & \text{if } \mathbf{x}' \text{ satisfies the check } i, \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

That is to say,

$$\begin{aligned} \Pr(s_i | \mathbf{x}') &= \delta \left(\sum_{j' \in V_1} (s_{ij'} | x'_{j'}), s_i \right) \\ &= \delta \left(\sum_{j' \in \mathcal{N}(i) \setminus \{j\}} (s_{ij'} | x'_{j'}), s_i - (s_{ij} | \alpha) \right) \end{aligned} \quad (14)$$

where

$$\delta(x, y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

is the Kronecker delta.

For QLDPC stabilizer codes, Eq. (12) can be computed efficiently using a fast-Fourier-transform-like recursive iteration. In other words, we observe that

$$R_{ij}^\alpha = R_{ij; \mathcal{N}(i) \setminus \{j\}, s_i - (s_{ij} | \alpha)} \quad (16)$$

where

$$R_{ij; J, b} = \sum_{\{x'_{j'} : j' \in J\}} \left[\delta \left(\sum_{j' \in J} (s_{ij'} | x'_{j'}), b \right) \prod_{j' \in J} Q_{ij'}^{x'_{j'}} \right] \quad (17)$$

for all $b \in GF(p)$. Then we can evaluate Eq. (12) by recursively applying the identity

$$R_{ij; J, b} = \sum_{c \in GF(p)} R_{ij; J_1, c} R_{ij; J_2, b-c} \quad (18)$$

for any partition $\{J_1, J_2\}$ of the set J with $|J_1| \approx |J_2|$ until $|J| = 1$. (And surely for $J = \{j'\}$, $R_{ij; J, b}$ can be calculated directly using Eq. (16).)

After computing R_{ij}^α efficiently, each check node s_i sends the message R_{ij}^α to the variable node x_j for all $j \in \mathcal{N}(i)$. Next, each variable node updates the messages

$$Q_{ij}^\alpha = \phi_{ij} f_j^\alpha \prod_{i' \in \mathcal{M}(j) \setminus \{i\}} R_{i'j}^\alpha \quad (19)$$

according to the information $R_{i'j}^\alpha$'s from check nodes $s_{i'}$'s for all $i' \in \mathcal{M}(j) \setminus \{i\}$, where $\mathcal{M}(j) \equiv \{i : H_{ij} \neq 0\}$ is the set of checks involving variable node x_j . The normalization constants ϕ_{ij} 's ensure that the sum of conditional probabilities $\sum_{\alpha \in GF(q^2)} Q_{ij}^\alpha = 1$.

After each round of message passing, we compute the pseudo-posterior probabilities

$$Q_j^\alpha = \phi_j f_j^\alpha \prod_{i \in \mathcal{M}(j)} R_{ij}^\alpha, \quad (20)$$

where ϕ_j is a normalization constant making $\sum_{\alpha} Q_j^\alpha = 1$. We now set \tilde{x}_j , the j th component of the tentative decoding \tilde{x} , to α if $Q_j^\alpha \geq Q_j^\beta$ for all $\beta \in GF(q^2)$. And we denote this operation by

$$\tilde{x}_j = \underset{\alpha \in GF(q^2)}{\operatorname{argmax}} Q_j^\alpha. \quad (21)$$

The decoding algorithm iterates until either the tentative decoding \tilde{x} is consistent with the observed syndrome (that is, $s_i = \sum_{j \in V_1} (H_{ij} | \tilde{x}_j)$ for all $i \in V_2$) or a pre-determined maximum rounds of message passing is reached.

To summarize, the belief propagation algorithm can be applied to decode QECC codes because its decisions depend

only on our prior assumptions of the noise of the channel and the measurement results for an independent noise channel of the error syndrome. Moreover, it decodes QLDPC codes efficiently partly because each summand in Eq. (12) can be expressed as a sum of products.

C. Our protocol

After all the above preliminary discussions, we now report our adaptive entanglement distillation scheme \mathfrak{P}_{BP} , which is an EDP2 protocol using (binary) QLDPC codes to distill EPR pairs.

[The Adaptive Entanglement Distillation Scheme \mathfrak{P}_{BP}]

- 1) Alice prepares n copies of EPR pairs $|\Psi^+\rangle \equiv (|00\rangle + |11\rangle)/\sqrt{2}$ and sends the second half of each pair to Bob through a noisy channel. Alice and Bob set the level ℓ to 1.
- 2) Alice and Bob measure their corresponding shares of the noisy EPR pairs using a pre-determined QLDPC code with a sparse parity check matrix $H[\ell]$ with the help of (unentangled) ancillas. Alice sends her measurement results to Bob. And then Bob computes the error syndrome $s[\ell](e)$, where e is the noise vector of the state they shared.
- 3) Using the belief propagation algorithm and Eq. (20), Bob computes the posterior marginal probabilities $Q_j^\alpha[\ell]$ that his j th qubit has experienced an error $\alpha \in GF(4)$ given the messages passed from all its check nodes. From the posterior marginal probabilities, Bob deduces a tentative decoding $\tilde{x}[\ell]$ based on the measured error syndrome $s[\ell](e)$.
- 4) If a tentative decoding $\tilde{x}[\ell]$ satisfying $H[\ell]\tilde{x}[\ell] = s[\ell](e)$ is found within the first m_{\max} rounds of message passing, then $\tilde{x}[\ell]$ is also a self-consistent error vector. In this case, what Bob needs to do is to perform the error correction by applying the additive inverse of the pseudo-posterior noise vector, namely $-\tilde{x}[\ell]$, to his qubits. Finally, Alice and Bob finish up by running the encoding circuit for $H[\ell]$ backward to distill copies of almost perfect EPR pair. (See Fig. 1a.) This marks the end of our scheme.
- 5) If $H[\ell]\tilde{x}[\ell] \neq s[\ell](e)$ even after m_{\max} rounds of belief propagation message passing, then Alice and Bob discard those EPR pairs whose believes of finding valid decodings are low. More precisely, they fix an entropy threshold $h_{th}[\ell]$ and throw away the j th EPR pair if the entropy of the pseudo-posterior probabilities

$$\begin{aligned} & h_4(Q_j[\ell]) \\ &= h_4(\{Q_j^\alpha[\ell] : \alpha \in GF(4)\}) \\ &= - \sum_{\alpha \in GF(4)} Q_j^\alpha[\ell] \log_2 Q_j^\alpha[\ell] \end{aligned} \quad (22)$$

is greater than $h_{th}[\ell]$.

The detailed procedure to throw away a EPR pair requires attention. According to the belief propagation algorithm, Alice and Bob believe that the most probable error experienced by the j th EPR pair is $\alpha_j[\ell] = \{\alpha[\ell] \in$

$GF(4) : Q_j^\alpha[\ell] \geq Q_j^\beta[\ell], \forall \beta \in GF(4)\}$. So Bob first apply $-\alpha_j[\ell]$ to his share of the j th EPR pair. Surely, there are more than one possible encoding circuit for $H[\ell]$ and running any of these encoding circuit backward can correctly decode $H[\ell]$ in the absence of noise. Since the tentative decoding cannot be found, in order to minimize the decoding error, Alice and Bob run the encoding circuit backward in which the sum of the entropies of the pseudo-posterior probabilities for the message qubits are minimized. After applying this decoding circuit, they can throw away those shared EPR pairs with high entropy of the pseudo-posterior probabilities. (See Fig. 1.)

- 6) Alice and Bob increase the level ℓ by 1. If it exceeds a pre-determined number ℓ_{\max} , they give up all their shared particles and start over again. Otherwise, they construct another sparse parity check matrix $H[\ell]$ orthogonal to $H[1], H[2], \dots, H[\ell-1]$ with respected to the skew-symmetric inner product in Eq. (7). In general, the choice of $H[\ell]$ may depend on the marginal posterior probabilities $Q_j^\alpha[1]$'s, $Q_j^\alpha[2]$'s, \dots , $Q_j^\alpha[\ell-1]$'s. They continue the decoding by going back to step 2.

Three remarks are in placed. First, for a sufficiently low channel error rate, a self-consistent error vector is likely to be found without throwing away any EPR pair in step 5. Besides, this self-consistent vector is equal to the noise vector e . Consequently, our protocol \mathfrak{P}_{BP} is reduced to a QECC-based scheme. While for a sufficiently high channel error rate, a self-consistent error vector is unlikely to be found. Together with suitable choices of the entropy thresholds $h_{th}[\ell]$'s, our protocol \mathfrak{P}_{BP} becomes, in effect, an EDP2 based scheme. In this respect, a row of $H[\ell]$ may be used either for error recovery or error rejection depending on the error syndrome measurement results. This fulfills our goal of finding an adaptive entanglement distillation scheme that estimates the error rate and performs the necessary error recovery or error rejection simultaneously. Second, our scheme can be generalized to distilling generalized Bell states readily. Third, $H[\ell]$'s should be picked in such a way that their error correcting capabilities increase as the number of levels ℓ increases; and we report a simple adaptive way to do so efficiently in the coming subsection.

D. The choice of parameters for \mathfrak{P}_{BP} and the performance indicators

In this pilot study, we only consider the performance of our scheme for a depolarizing channel, namely, each EPR pair has an equal and independent chance of experiencing a Pauli error. We denote the probability that a qubit experience any one of the Pauli errors by p_0 . Moreover, we do not focus on the performance of a particular QLDPC code used in our EDP2 protocol. Instead, we consider the average performance over an ensemble of QLDPC codes used. Moreover, these codes are randomly selected using the method reported in Sec. III-A.

1) *The choice of parameters for \mathfrak{P}_{BP} :* The scheme \mathfrak{P}_{BP} involves the use of QLDPC codes and an adaptive procedure according to the error syndrome measurement results. We

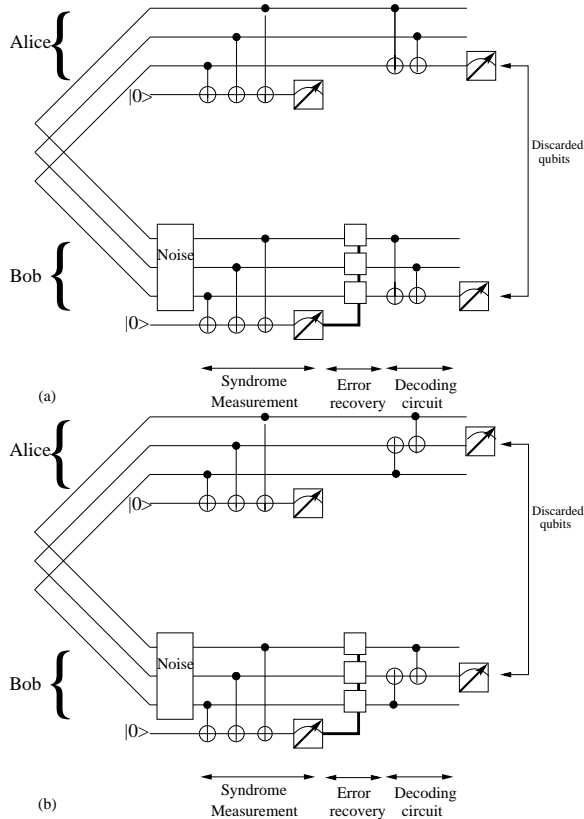


Fig. 1. An illustrative example of \mathfrak{P}_{BP} using $H[1] = (\omega_4 \ \omega_4 \ \omega_4)$. In the absence of noise, decoding circuits in (a) and (b) are equivalent up to permutation of entangled qubits. But in the presence of noise, their performances may differ due to the error propagation in the decoding process.

try to understand the effects of these two ingredients on the performance by studying two implementations of \mathfrak{P}_{BP} .

Implementation A:

- (i) We fix the maximum number of levels $\ell_{\max} = 1$.
- (ii) We choose the QLDPC code $H[1]$ using the extended bicycle construction reported in Sec. III-A.
- (iii) We set the entropy threshold $h_{\text{th}}[1]$ to

$$\begin{aligned} h_{\text{th}} &\equiv h_{\text{th}}[1] = S(W_{p_0}) \\ &= -(1-p_0) \log(1-p_0) - p_0 \log \frac{p_0}{3} \end{aligned} \quad (23)$$

where

$$\begin{aligned} W_{p_0} &= \frac{1-p_0}{3} |\Psi^+\rangle\langle\Psi^+| + \frac{p_0}{3} (|\Psi^-\rangle\langle\Psi^-| + \\ &\quad |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-|) \end{aligned} \quad (24)$$

is the Werner state. The rationale behind this choice is that after passing through the depolarizing channel with quantum error rate p_0 , the density matrix $|\Psi^+\rangle\langle\Psi^+|$ becomes the Werner state W_{p_0} . Therefore, in the absence of any additional information on the errors syndrome measurement, the entropy of the uncertainty of the kind of error experienced by each $|\Psi^+\rangle$ is equal to W_{p_0} . Thus, this choice of entropy threshold means that a pair is discarded only if a self-consistent tentative decoding $\tilde{x}[1]$ cannot be found and the belief propagation algorithm is unable to

improve Bob's knowledge on the kind of error the pair has experienced.

- (iv) We put the maximum number of rounds of message passing $m_{\max} \approx 10$. In fact, further increasing m_{\max} does not improve the performance of message passing algorithm to correctly find out the noise vector. (See, for example, Ref. [31]).

Clearly, Implementation A can be used to study the performance of EDP2 using QLDPC codes without an adaptive procedure. To study the power of adaptation, we consider Implementation B below.

Implementation B:

- (i) We fix $\ell_{\max} = n/4$ where n is the codeword size of $H[\ell]$'s.
- (ii) We first pick a QLDPC code H using the extended bicycle construction. From this H , we build a hierarchy of codes as follow. The first layer contains one code, namely, H itself. The u th layer of codes are those formed by deleting exactly u rows from H . Moreover, a layer u code H_u is said to be connected to a layer $(u+1)$ code H_{u+1} if H_{u+1} can be formed by removing one row from H_u . Now, we randomly pick a layer ℓ_{\max} code in this hierarchy to be our $H[1]$, namely, the QLDPC code used in the first level of decoding in \mathfrak{P}_{BP} . If the tentative decoding is not matched, that is, $H[1]\tilde{x}[1] \neq s[1](e)$, then the code $H[2]$ used in the second level of decoding in \mathfrak{P}_{BP} is selected among those layer $(\ell_{\max} - 1)$ codes in the hierarchy that are connected to $H[1]$. Surely, such a choice should, as far as possible, maximize the belief on the errors experienced by the EPR pairs after running the belief propagation algorithm with the code $H[2]$. More precisely, $H[2]$ is chosen by adding one row of the parity check matrix H that is not present in $H[1]$. And this additional row is selected so as to maximize $\sum_{j \in V_2} h_4(Q_j[\ell = 1])$, where $Q_j[\ell = 1]$ is the pseudo-posterior probability obtained by running the belief propagation algorithm with the code $H[1]$. The codes $H[3], \dots, H[\ell_{\max}]$ are picked in a similar manner until either a consistent tentative decoding is found or when $\ell = \ell_{\max}$. In this way, we construct a sequence of mutually orthogonal QLDPC codes adaptively and effectively.
- (iii) We set $h_{\text{th}}[\ell_{\max}] = S(W_{p_0})$. More importantly, we put $h_{\text{th}}[\ell] = 2$ for all $\ell < \ell_{\max}$, namely, the maximum possible value for a noisy EPR pair. In this way, we avoid throwing away noisy EPR pairs prematurely.
- (iv) As in Implementation A, we set $m_{\max} \approx 10$.

The choice of ℓ_{\max} in Implementation B requires clarification. In order to fully utilize the adaptive nature of \mathfrak{P}_{BP} , ℓ_{\max} should not be too small. Nevertheless, the average number of checks per variable node for $H[1]$ will be less than 1 if ℓ_{\max} is about $n/2$, making $H[1]$ useless for distillation. In what follows, we take the middle path by fixing $\ell_{\max} = n/4$.

2) *Performance indicator*: The yield D is used as the performance indicator. It is defined as the expected number of input pairs needed per output perfect EPR pair in the limit of a large number of input pairs. The yield is high if the rate of the quantum code used is high and the decoder error rate, namely, the chance for a qubit to be incorrectly decoded is low. We use the symbols D_A and D_B to denote the average yields of our EDP2 protocol \mathfrak{P}_{BP} using Implementations A and B over the ensemble of QLDPC codes, respectively.

IV. PERFORMANCE OF \mathfrak{P}_{BP}

We study the performance of \mathfrak{P}_{BP} by numerical simulations. Actually, our simulations show that the yields D_A and D_B depend chiefly on the values of d_v and d_c for the codes $H[1]$ and H in Implementations A and B, respectively. In other words, the yields are not sensitive to the actual sparse vector (α_i) used in the extended bicycle code construction. In all the figures below, each data point represents the average yield D_{BP} for \mathfrak{P}_{BP} over 1000 independently generated noise vectors. The associated one-sigma-level error bar is also shown.

Our simulations show that, within $\approx 0.1\sigma$ level of uncertainty, the yield D of Implementations A and B do not depend on the codeword size n provided that $n \gg d_v, d_c$. (See Fig. 2.) So, we fix the value of $n = 960$ in all our subsequent discussions.

Another general feature concerning the error bars requires explanation. As shown in Fig. 2, the sizes of error bars in a typical D against p_0 plot depend strongly on the value of p_0 . When p_0 is sufficiently small, the tentative decoding obtained from \mathfrak{P}_{BP} correctly predicts the errors experienced by the EPR pairs most of the time. Hence, the error bar size is small. When p_0 is so high that it cannot be handled by the QLDPC code used, most of the EPR pairs will be thrown away after running \mathfrak{P}_{BP} . This makes both the yield D and the size of its error bar small. Interestingly, when p_0 is in between these two extremes, the probabilities of correctly and incorrectly finding the tentative decoding are comparable. More importantly, these two cases have drastically different yields. As a result, the error bar of the average yield D in this regime is very large. In other words, the large variance of D in this regime is intrinsic and is not the result of insufficient sampling.

Let us compare the performances of Implementation A using $(2, d_c)$ -regular QLDPC codes with the recurrence method [5] and its extension by Leung and Shor [2]. This comparison makes sense because of two reasons. First, the recurrence method makes use of $(1, 2)$ -regular quantum codes with block size 2; and the Leung and Shor method makes use of $(2, 4)$ -regular ones with block size 4. Besides, all these three methods use quantum codes with minimum distance 2. In other words, these codes can only detect but cannot correct quantum errors. As shown in Fig. 3, the maximum error tolerable rate for Implementation A using $(2, 4)$ -regular QLDPC codes is higher than both the recurrence and the Leung and Shor methods provided that the channel error rate $p_0 > 0.28$. (In fact, our scheme can tolerate up to at least $p_0 = 0.30$.) This result demonstrates the power of using QLDPC codes to distill very noisy EPR pairs using two-way classical communications. Fig. 3 also depicts that by

using $(2, d_c)$ -regular QLDPC codes with $d_c \geq 4$, the yield of Implementation A is higher than the recurrence method as well as the Leung and Shor's whenever $p_0 < 0.05$. This is not surprising because more EPR pairs are sampled in each error syndrome measurement as d_c increases. Nonetheless, it also makes the yield decrease for a large value of p_0 because propagation of quantum errors due to decoding in step 5 of \mathfrak{P}_{BP} is more serious for a large d_c .

Figs. 4a and 4b show the yields D_A using (d_v, d_c) -regular QLDPC codes for different d_v at a fixed quantum code rate of $1 - d_v/d_c = 1/2$. As d_v increases, the distance of the QLDPC code $H[1]$ increases. That is why the D_A against p_0 curve is strictly decreasing for $d_c \lesssim 5$, indicating that the code $H[1]$ is only error detecting. In contrast, this curve is flat for very small p_0 whenever $d_c \gtrsim 6$, indicating that the code $H[1]$ becomes error correcting. Note further that in the latter case, the yield D_A drops rapidly when p_0 increases beyond the flat region of the yield curve. This is a consequence of our error rejection mechanism. Recall that those EPR pairs which have the low belief on the kind of error experienced will be thrown away. Since d_v is small and $H[1]$ is sparse, statistical fluctuations may allow Alice and Bob to correctly identify a few non-erroneous pairs via the belief propagation algorithm. And these correctly identified pairs will be kept, making the error rate of the remaining pairs lower than that of the original n EPR pairs. In other words, by increasing d_v while keeping the ratio d_v/d_c fixed, it is harder to identify of these non-erroneous pairs as the Tanner graph associated with H becomes more connected. Furthermore, this increase in the connectivity of the Tanner graph implies that backward propagation of quantum errors as a result of error syndrome measurement is more serious. This is also a contributing factors to the low yield as d_v increases by keeping d_v/d_c fixed.

Fig. 5 shows the comparison of yields of our Implementation A and B. In line with our expectation, for the same set of parameters d_v , d_c and n , Implementation B outperforms Implementation A for small p_0 where their performances converge as p_0 increases. The adaptive nature of Implementation B allows Alice and Bob to pick a quantum code that is sufficiently powerful to combat the channel noise on the one hand and is sufficiently high rate to give a good yield on the other hand. This demonstrates the power of adaptation using \mathfrak{P}_{BP} . Nevertheless, adaption of this kind cannot improve the capacity of \mathfrak{P}_{BP} when p_0 is large. In fact, Fig. 5 shows that Implementations A and B can handle the same maximum error rate provided that $H[1] = H$. (More precisely, $D_B(p_0) \geq D_A(p_0)$, but $D_B(p_0) = 0$ if and only if $D_A(p_0) = 0$.) This finding is not completely surprising. When p_0 is sufficiently high, it is likely that tentative decoding for $H[1], H[2], \dots, h[\ell_{\max} - 1]$ cannot be found. Thus, in this regime, Implementation B is reduced to Implementation A.

In Fig. 6, we compare the performances of Implementation B using $(2, d_c)$ -regular QLDPC codes with the recurrence method [5] and Leung and Shor's protocol [2]. Using $(2, 8)$ -regular QLDPC codes, the yield of Implementation B is higher than that of the recurrence method as well as the the Leung and Shor's protocol whenever $p_0 < 0.07$ because at such small p_0 , there is no need to perform so heavy different parity checks.

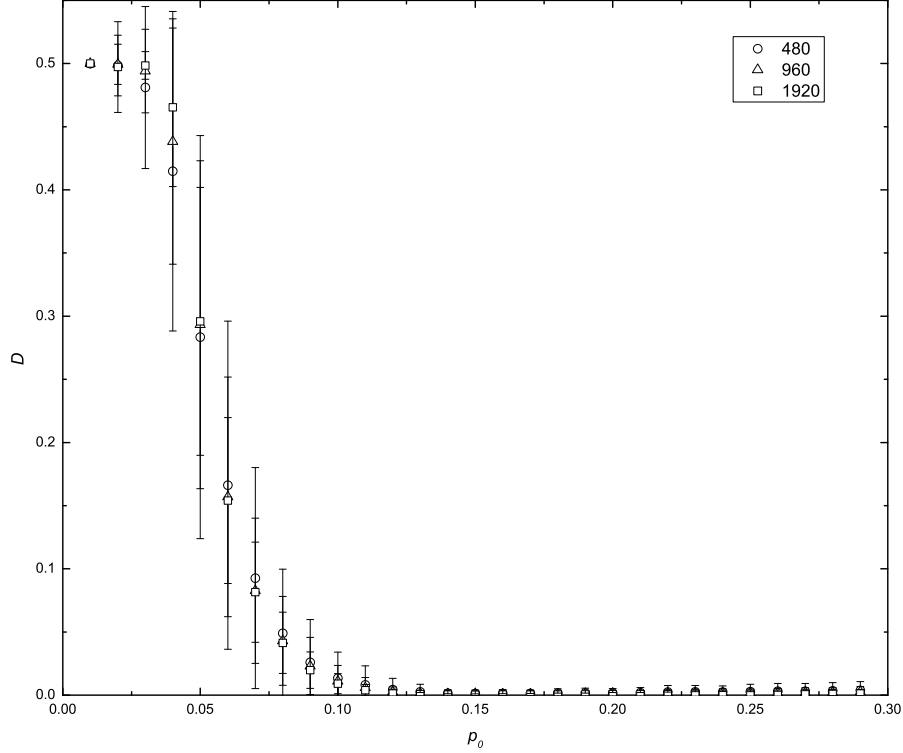


Fig. 2. A plot of the yield D_A of Implementation A using (8, 16)-regular QLDPC codes for different codeword sizes n against the error probability p_0 .

Similar to the case of Implementation A, we find that the maximum tolerable rate for Implementation B using (2, 4)-regular QLDPC codes is higher than both the recurrence and the Leung and Shor protocols provided that the channel error rate $p_0 > 0.28$.

V. CONCLUSIONS AND OUTLOOK

In conclusion, we have introduced an adaptive two-way entanglement distillation protocol \mathfrak{P}_{BP} using QLDPC codes and belief propagation decoding algorithm. In particular, we demonstrate the power of using QLDPC codes and/or adaptation in EDP2 protocol. Moreover, we find that the yield of our scheme \mathfrak{P}_{BP} using an ensemble of (2, 4)-regular QLDPC codes is higher than that of Leung and Shor [2], [3] for handling depolarization error whenever the error probability p_0 is greater than 0.28 or less than or equal to 0.07. In fact, using this choice of QLDPC codes, our scheme \mathfrak{P}_{BP} can tolerate depolarization errors up to at least a quantum error rate of 30%.

The high yield together with the reasonably high error tolerance capability of our scheme are due to several reasons. First, the scheme is adaptive in the sense that each parity check may be used for error rejection or error correction depending on the measurement results. In this way our scheme becomes an effectively one-way QECC-based scheme when the quantum error rate is low. And on other hand, it becomes a EDP2 based scheme when the quantum error rate is high.

Note that as long as Alice and Bob find that a certain qubit has experienced a certain quantum error with high probability,

they can apply the necessary error correction operation to recover a high fidelity EPR pair. What causes the trouble is that sometimes Alice and Bob are unable to correctly determine or have little confidence on the kind of errors has occurred in their shared qubits. This leads us to the second reason why our scheme \mathfrak{P}_{BP} is so efficient. This is due to the fact that the belief propagation decoding algorithm is able to efficiently find out the entropy of the kind of quantum errors believed to be experienced by which of the qubits are higher than the entropy threshold $h_{th}[\ell]$. Consequently, Alice and Bob may throw this kind of qubits away. The belief propagation approach is a Bayesian approach. It takes into account Alice and Bob's initial belief on the channel and the information obtained from the error syndrome measurements in a transparent way in computing the final belief of the errors occurred. By choosing a sufficiently long codeword size n , it is highly probable that the error rate of a few variable nodes that are connected to a check node in the corresponding Tanner graph is significantly lower than the system average. The belief propagation decoding algorithm can help Alice and Bob to identify these variable nodes. By selectively keeping this kind of variable nodes (that is, these qubits) for entanglement distillation, our EDP2 protocol \mathfrak{P}_{BP} is able to tolerate a reasonably high quantum error rate.

Finally, the last reason behind the good performance of \mathfrak{P}_{BP} lies in the use of QLDPC codes. It ensures that the average error correcting capability of its punctured code is still acceptable.

A number of followup researches along this line have to

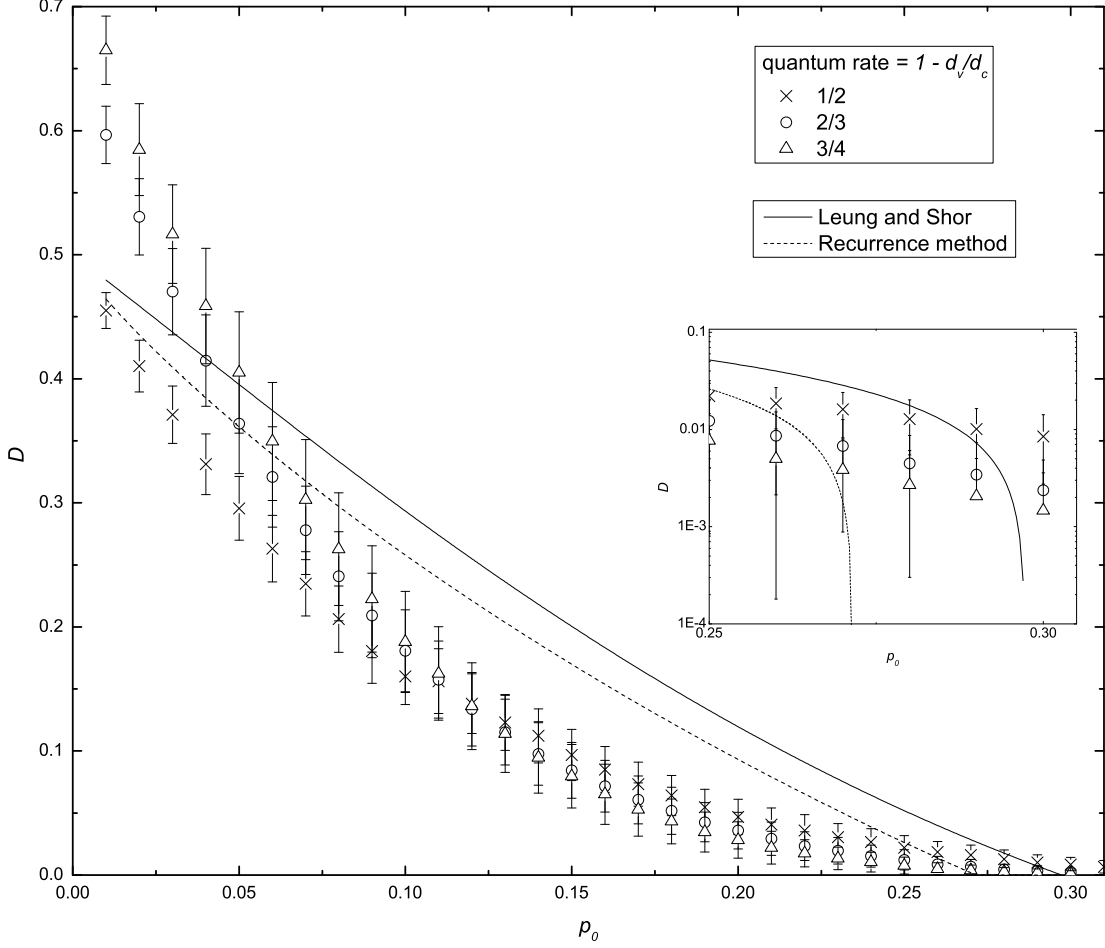


Fig. 3. A plot of the yield D_A of Implementation A using (d_v, d_c) -regular QLDPC codes for different quantum rates against the error probability p_0 . The values of d_v and n are fixed to be 2 and 960, respectively.

be done. For instance, the effect of the choice of the entropy thresholds $h_{\text{th}}[\ell]$ on the yield D_{BP} should be studied. More importantly, the choice of QLDPC codes with sparse parity check matrices $H[\ell]$'s in a multi-level (that is, $\ell_{\text{max}} > 1$) setting requires thorough investigations in order to understand the capability and the tradeoff between the yield and the maximum error tolerable rate of our protocol. In particular, we believe that the maximum error tolerable rate for \mathfrak{P}_{BP} can be pushed up further by adaptively concatenating QLDPC codes.

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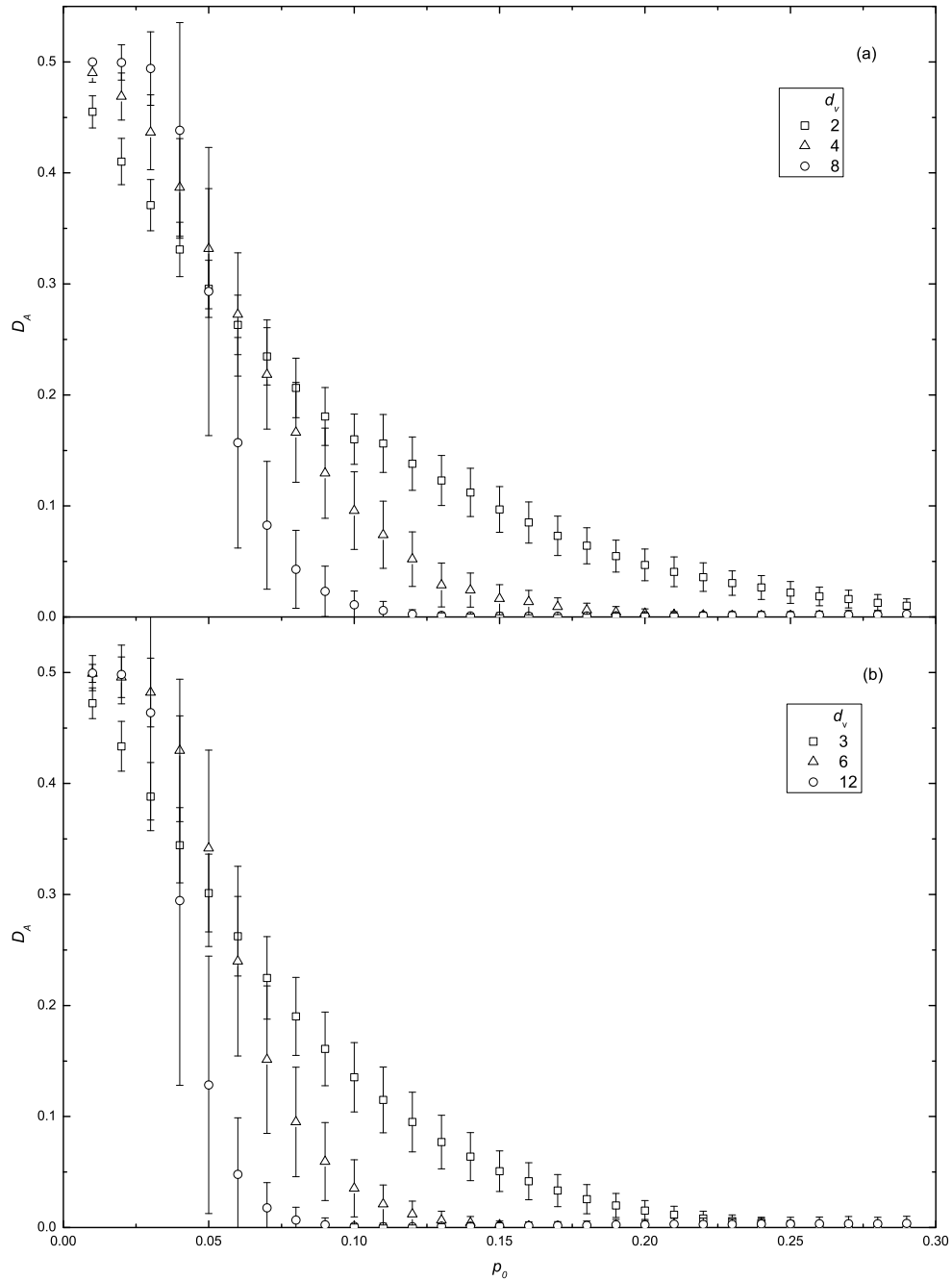


Fig. 4. A plot of the yield D_A of Implementation A for different d_v 's versus the error probability p_0 . The values of quantum rate and n are fixed to $1/2$ and 960 respectively.

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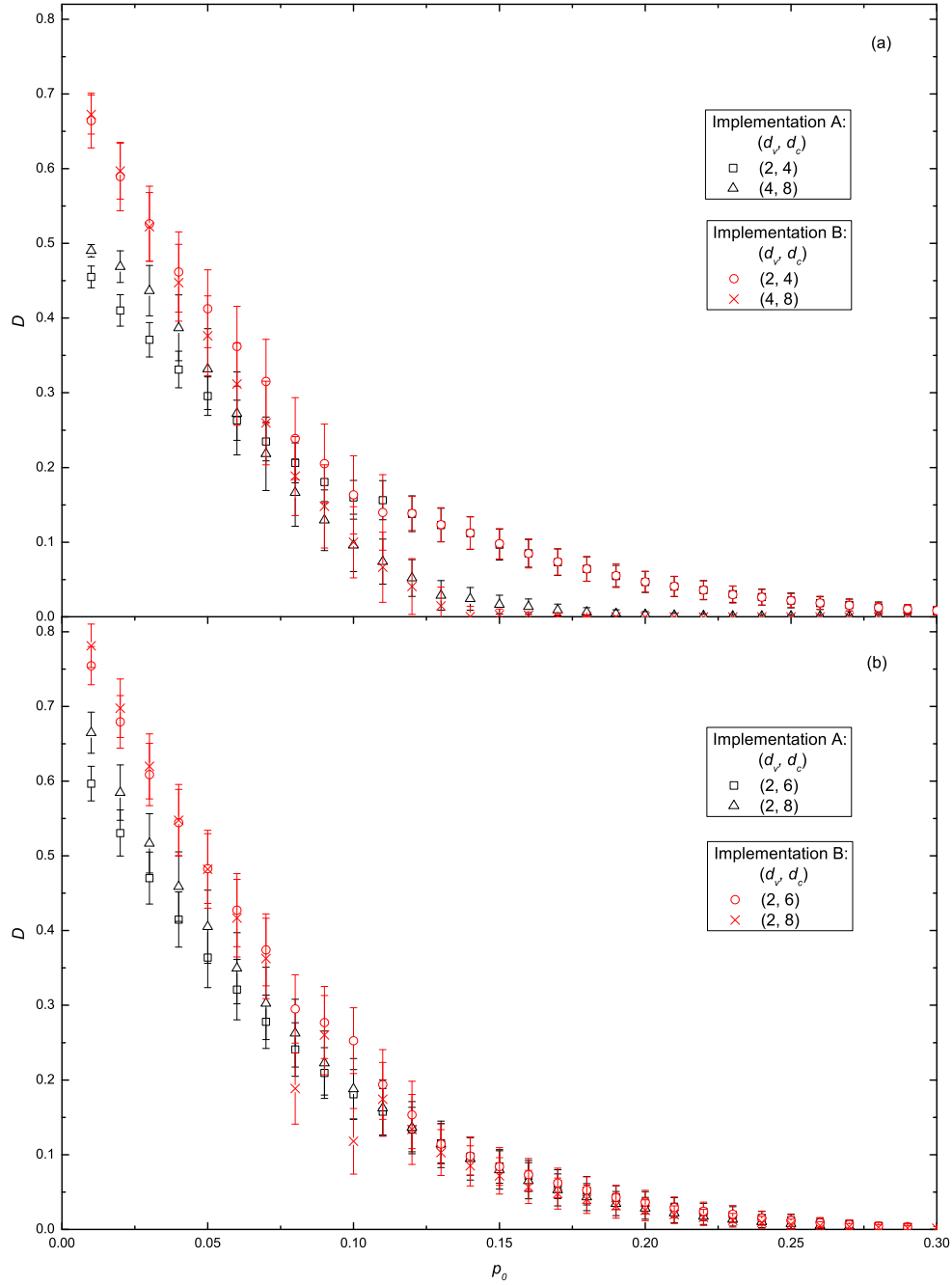


Fig. 5. Comparison of the yield D of Implementation A and B. (a) The codes are fixed at quantum rate $1/2$. (b) The d_v for codes are fixed at 2.

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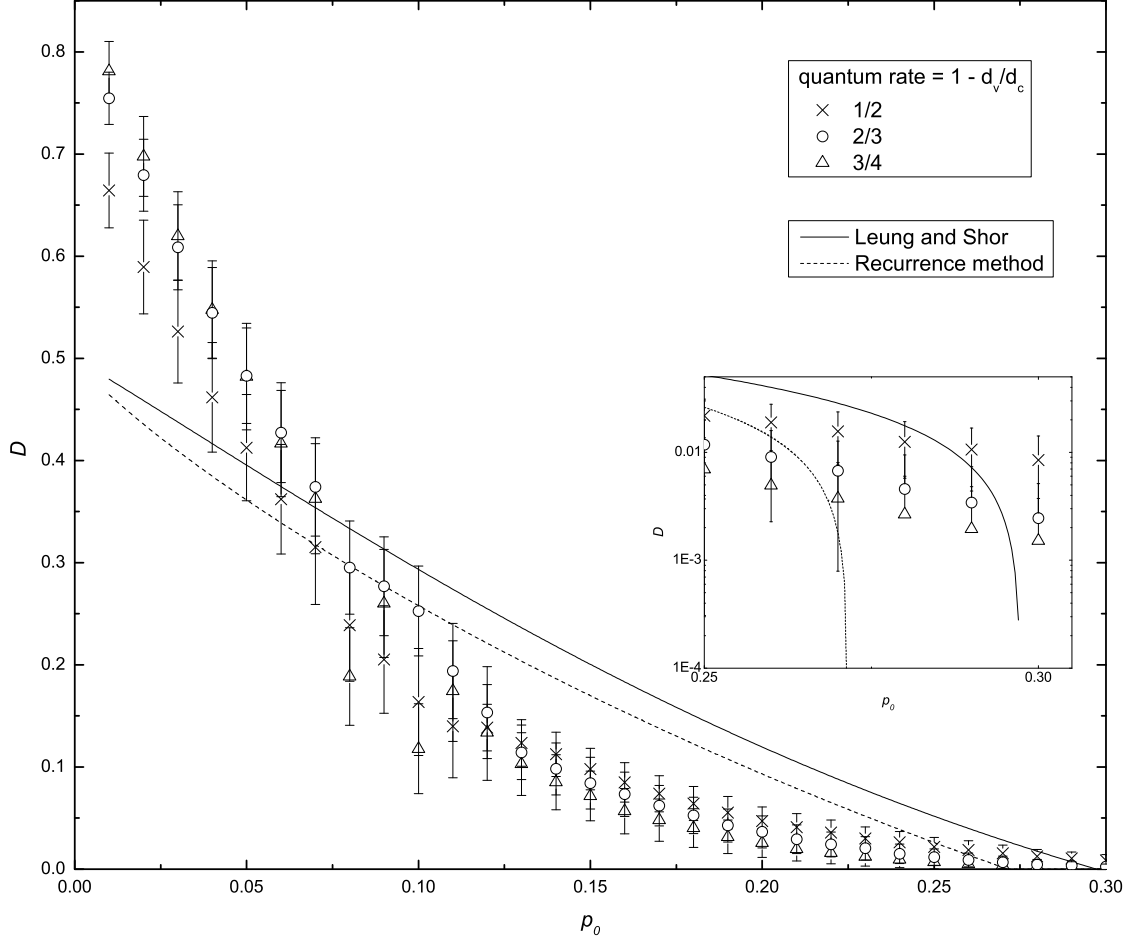


Fig. 6. A plot of the yield D_B of Implementation B using (d_v, d_c) -regular QLDPC codes for different quantum rates against the error probability p_0 . The values of d_v and n are fixed to be 2 and 960, respectively.